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A Note on k-Step Hamiltonian Graphs

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ABSTRACT

For a given integer k, a given graph G on n vertices is called k-step Hamiltonian (or just k-SH) if the vertices of G can be labeled as $v_1, v_2, ..., v_n$ such that $d(v_1, v_n) = k$ and $d(v_i, v_{i+1}) = k$ for each $i = 1, 2, ..., n-1$. In this paper, we present a construction namely B-construction that produces a $(k+i)$-SH graph from any k-SH graph G for every positive integer $i \geq 1$.

Keywords: Hamiltonian graph, k-Step Hamiltonian graph.

1. Introduction

We denote by $G = (V, E)$ a simple graph, where $V = V(G)$ is referred as the vertex-set and $E = E(G)$ as the edge-set. For any vertex v, the set $N_G(v) = \{u|uv \in E(G)\}$ is the *open neighborhood* of the vertex v and the set $N_G[v] = \{v\} \cup N_G(v)$ is the *closed neighborhood* of v. The number of neighbors of a vertex v in G is the *degree* of v, $\deg_G(v)$, (or just $\deg(v)$ if G is clear). For a pair of vertices u and v in G, the minimum length of a path from u to v (that is the minimum number of edges) is the *distance* from u to v, and we denote it by $d(u, v)$.

The maximum $d(u, v)$ among two vertices u and v of G is the *diameter* of G(denoted as $diam(G)$). Consider a set $\{a_1, a_2, \ldots, a_k\}$ of distinct integers such that $0 < a_1 < a_2 < \ldots < a_k \leq \lfloor n/2 \rfloor$. Two important class of graphs that play important roles in this paper are circulant graphs and corona graphs. A *circulant graph* $C_n(a_1, a_2, \ldots, a_k)$ is a graph of order n with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that v_i is adjacent to v_{i+a_j} for all $a_j \in \{a_1, a_2, \ldots, a_k\}$, where subscripts are to be read modulo n. The *corona graph* of G is a graph obtained from the graph G by adding a leaf to every vertex of G. The corona graph of G we denote by $cor(G)$. For other notations and terminologies not given here, we refer to standard graph theory books.

A graph $G = (V, E)$ is called *Hamiltonian* if there exists a spanning cycle. For a recent development and open problems related to Hamiltonicity of graphs, Gould (2003) has an excellent survey.

Lau et al. (2014b) extended the concept of Hamiltonicity to k-step Hamiltonicity. They introduced the concept of $AL(k)-$traversal followed by k-step Hamiltonian graph as follows: For $k \geq 1$, a graph G of order n is said to admit an $AL(k)-$traversal if we can arrange vertices of G as the sequence of vertices $v_1, v_2, \ldots, v_n$ s.t. $d(v_i, v_{i+1}) = k$ for $i = 1, 2, \ldots, n-1$. A graph G is $k-$step Hamiltonian (or just $k-$SH) if it has an $AL(k)-$traversal and $d(v_1, v_n) = k$. Then, the sequence $v_1, v_2, \ldots, v_n, v_1$ is called a $k-$step Hamiltonian walk of G. Clearly, k-SH graphs with $k = 1$ are the Hamiltonian graphs.

The concept of k-SH graphs has been further studied in, for example, Abd Aziz et al. (2018), Ho et al. (2016), Lau et al. (2015, 2014a) and Lee and Su (2016). Several classes of $k-$SH graphs including tripartite graphs, cycles, grid graphs, cubic graphs and subdivision graphs of cycles, have been studied, see Ho et al. (2016), Lau et al. (2015, 2014a,b), Lee and Su (2016). In Abd Aziz et al. (2018), we give a construction that produces an infinite family of k-SH graphs from any given k-SH graph G. By now, there is no construction to produce a $(k+1)$-SH graph from any k-SH graph G. Such construction for some other graph parameters are known, see for example, Mycielski's construction for coloring.

In this paper, we first study k-SH circulant graphs $C_n(1,2)$. We then present a construction namely B-construction that produces a $(k+1)$-SH graph from any k-SH graph G. Our construction thus produces a $(k+i)$-SH graph from any k-SH graph G for every positive integer $i \geq 1$.

2. Preliminary results

In this section we study k-SH circulant graphs $C_n(1,2)$.

Theorem 2.1. *If $gcd(n, 2j-1) = 1$ for $n \geq 6$ and $2 \leq j \leq \lceil \frac{n-1}{4} \rceil$, then $C_n(1,2)$ is $j-$SH.*

Proof. Let $G = C_n(1,2)$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Note that $diam(G) = \lceil \frac{n-1}{4} \rceil$. With the values of n and j given in the hypothesis, we show that the sequence of vertices $v_1, v_{1+(2j-1)}, \ldots, v_{1+(n-1)(2j-1)}, v_1$ is a $j-$step Hamiltonian walk for G. For this purpose, we first need to show that $\{1, 1+(2j-1), 1+2(2j-1), \ldots, 1+(n-1)(2j-1)\}$ (mod n) is a set of distinct integers. Suppose that $v_{1+a(2i-1)} = v_{1+b(2i-1)}$ for $0 \leq a < b < n$. This implies that $a(2j-1) \equiv b(2j-1)(\text{mod } n)$. Since $gcd(n, 2j-1) = 1$, we have $a \equiv b(\text{mod } n)$ which also means $b \equiv a(\text{mod } n)$. But, since $b-a$ is a positive integer and $b-a < n$, n does not divide $b-a$ and so b is not congruent to a modulo n, a contradiction. Thus, the integers in the above set are distinct. We next show that the distance between any two consecutive vertices in the sequence is j. From the construction of graph G, v_i is adjacent to both v_{i+1} and v_{i+2} and thus $d(v_i, v_{i+1}) = d(v_i, v_{i+2}) = 1$ for $i = 1, 2, \ldots, n$ where subscripts are taken modulo n. Since $v_{1+(i+1)(2j-1)} = v_{1+i(2j-1)+(2j-1)}$ for $i = 0, 1, 2, \ldots, n-2$ and $j \leq diam(G)$, it is clear that $d(v_{1+i(2j-1)}, v_{1+(i+1)(2j-1)}) = j$. To complete the proof, we need to show that the last vertex in the sequence, (i.e. $v_{1+(n-1)(2j-1)}$) is at distance j to the initial vertex v_1. It suffices to show that $v_{1+(n-1)(2j-1)+2(j-1)} = v_n$. Expanding the index of the vertex, we get

$$1 + n(2j-1) - (2j-1) + 2(j-1) \equiv 0 \equiv n \text{ (mod } n).$$

Therefore, the above sequence is indeed a $j-$step Hamiltonian walk in G, as desired. □

The next corollary follows immediately.

Corollary 2.1. *If $gcd(n, 2i-1) = 1$ for $2 \leq i \leq j$, then $C_n(1,2)$ is $i-SH$.*

3. Constructions

The following A-construction given in Abd Aziz et al. (2018) produces an infinite family of $k-$SH graphs from any given $k-$SH graph G.

A-Construction: For any graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, A-Construction produces a graph $A(G)$ with $V(A(G)) = \{v_1, v_2, \ldots, v_n\} \cup \{u_i : i = 1, 2, ..., n\}$ and $E(A(G)) = E(G) \cup \{u_i v_j : v_j \in N_G(v_i)\}$. Furthermore, we can define $A^m(G)$, the m-th iterated construction A of G for any $m \geq 1$, recursively by $A^1(G) = A(G)$, $A^2(G) = A(A(G))$, and $A^m(G) = A(A^{m-1}(G))$ if $m \geq 2$.

Theorem 3.1 (Abd Aziz et al. (2018))**.** *If G is a $k-SH$ graph, where $k \geq 2$ is an integer, then $A^m(G)$ is a $k-SH$ graph for all $m \geq 1$.*

We present here a new construction namely B-construction that produces a $(k+1)$-SH graph from any k-SH graph G.

B-Construction. Let G be a $k-$SH graph of order n, where $k \geq 1$ is an integer. Let $v_1, v_2, \ldots, v_n, v_1$ be a $k-$step Hamiltonian walk, and let v_i' be the leaf in $cor(G)$ adjacent to v_i, for $i = 1, 2, ..., n$. Then the B-construction produces a graph $B(G)$ from G as follows:

(i) For odd n, $B(G) = cor(G)$.

(ii) For even n, $B(G)$ is obtained from $cor(G)$ by the following scheme:

Step 1. For an integer m, $m \geq 6$ and $k \leq \lceil \frac{m-1}{4} \rceil - 1$ with $gcd(m, k+1) = 1$, the circulant graph $C_m(1,2)$ is $(k+1)-$SH by Theorem 2.1. Let $C_m^1(1,2)$ and $C_m^2(1,2)$ be two copies of $C_m(1,2)$. We label the vertices of a $(k+1)-$step Hamiltonian walk of $C_m^1(1,2)$(respectively $C_m^2(1,2)$) by $u_{1,1}, u_{1,2}, \ldots, u_{1,m}, u_{1,1}$(respectively $u_{2,1}, u_{2,2}, \ldots, u_{2,m}, u_{2,1}$). Note that $d(u_{i,j}, u_{i,j+1}) = k+1$ for $i = 1, 2$ and for $j = 1, 2, \ldots, m$, where $j+1$ is taken modulo m.

Step 2. Identify the vertices $u_{1,1}, u_{1,m}, u_{2,1}$ and $u_{2,m}$ to the vertices v_n', v_1', v_n and v_1, respectively.

We can also define recursively $B^m(G)$, the m-th iterated construction B of G for any $m \geq 1$, by $B^1(G) = B(G)$, $B^2(G) = B(B(G))$, and $B^m(G) = B(B^{m-1}(G))$ if $m \geq 2$. The construction of graph $B(G)$ is shown in Figure 1.

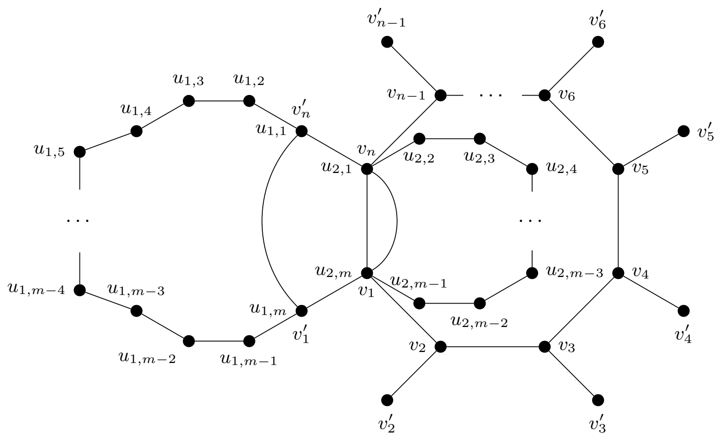

Figure 1: A graph $B(G)$ constructed from a $k-$SH graph G when n is even.

Theorem 3.2. *If G is $k-$step Hamiltonian, then $B(G)$ is $(k+1)-$step Hamiltonian.*

Proof. Consider a $k-$SH graph G. Let $v_1v_2...v_nv_1$ be a k-step Hamiltonian walk in G. Assume that n is odd. Let v'_i be the leaf adjacent to v_i in $cor(G)$, for $i = 1, 2, ..., n$. Note that $d(v_i, v'_i) = 1$ for $i = 1, 2, \ldots, n$ and $d(v_j, v_{j+1}) = k$ for $j = 1, 2, \ldots, n$ where $j+1$ is in modulo n. Then the sequence of vertices $v_1, v'_2, v_3, v'_4, \ldots, v'_{n-1}, v_n, v'_1, v_2, \ldots, v_{n-1}, v'_n, v_1$ is clearly a $(k+1)-$step Hamiltonian walk of $cor(G) = B(G)$. Consequently, $B(G)$ is $(k+1)$-SH. Next assume that n is even. Then we find that $B(G)$ is obtained from $cor(G)$ using two copies $C^1_m(1,2)$ and $C^2_m(1,2)$ of a circulant graph $C_m(1,2)$ where $m \geq 6$ and $k \leq \lceil \frac{m-1}{4} \rceil - 1$ such that $gcd(m, k+1) = 1$ as described in the construction. Note that $v_n = u_{2,1}$, $v'_n = u_{1,1}$, $v_1 = u_{2,m}$ and $v'_1 = u_{1,m}$ and so we have $d(v'_n, u_{1,2}) = d(u_{1,m-1}, v'_1) = d(v_n, u_{2,2}) = d(u_{2,m-1}, v_1) = k+1$. Then, $v_1, v'_2, v_3, v'_4, \ldots, v_{n-1}, v'_n, u_{1,2}, u_{1,3}, \ldots, u_{1,m-1}, v'_1, v_2, v'_3, v_4, v'_5, \ldots, v'_{n-1}, v_n,$ $u_{2,2}, u_{2,3}, \ldots, u_{2,m-1}, u_{2,m} = v_1$ is a $(k+1)$-step Hamiltonian walk in $B(G)$. Consequently, $B(G)$ is $(k+1)$-SH. □

Corollary 3.1. *Assume that G is a $k-$SH graph for some $k \geq 1$. For all $m \geq 1$, $B^m(G)$ is a $(k+m)$-SH graph.*

Corollary 3.1 indicates that from any Hamiltonian graph G_1, we can obtain an infinite chain of graphs $G_1, G_2, G_3, ...$ such that for any $j \geq 1$, G_j is j-SH.

Corollary 3.2. *Any graph G can be considered as a subgraph of a k-SH graph for any given $k \geq 1$.*

Proof. Consider a graph G. If G is Hamiltonian i.e. $1-$SH, then directly apply Corollary 3.1 to G. Otherwise, we add some edges to G that will result in a graph G' having a Hamiltonian cycle. Then apply Corollary 3.1 to G'. □

4. Conclusion

New construction namely B-construction is presented. The B-construction starts from any k-SH graph G and produces a $(k+1)$-SH graph. There are a lot of problems on k-SH graphs that yet to be settled. Among others, we propose the following.

Problem (I): Study coloring, domination and independence in k-SH graphs.

Problem (II): Study algorithmic aspects in k-SH graphs. It seems to us that the decision problem for k-SH graphs is NP-complete for many small classes of graphs.

Problem (III): Study k-SH graphs that the after removal of an edge they are not k-SH. This can be considered by removal of a vertex. Can you characterize such graphs?

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